Exercise 5G

1 a $(x_1, y_1) = (0, 1), (x_2, y_2) = (6, 9)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(6-0)^2 + (9-1)^2}$$
$$= \sqrt{6^2 + 8^2}$$
$$= \sqrt{36+64}$$
$$= \sqrt{100}$$
$$= 10$$

b $(x_1, y_1) = (4, -6), (x_2, y_2) = (9, 6)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(9-4)^2 + (6-(-6))^2}$$
$$= \sqrt{5^2 + 12^2}$$
$$= \sqrt{25+144}$$
$$= \sqrt{169}$$
$$= 13$$

c $(x_1, y_1) = (3, 1), (x_2, y_2) = (-1, 4)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1-3)^2 + (4-1)^2}$$
$$= \sqrt{(-4)^2 + 3^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

d $(x_1, y_1) = (3, 5), (x_2, y_2) = (4, 7)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-3)^2 + (7-5)^2}$$
$$= \sqrt{1^2 + 2^2}$$
$$= \sqrt{1+4}$$
$$= \sqrt{5}$$

e $(x_1, y_1) = (0, -4), (x_2, y_2) = (5, 5)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-0)^2 + (5-(-4))^2}$$
$$= \sqrt{5^2 + 9^2}$$
$$= \sqrt{25+81}$$
$$= \sqrt{106}$$

f $(x_1, y_1) = (-2, -7), (x_2, y_2) = (5, 1)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-(-2))^2 + (1-(-7))^2}$$
$$= \sqrt{(5+2)^2 + (1+7)^2}$$
$$= \sqrt{7^2 + 8^2}$$
$$= \sqrt{49+64}$$
$$= \sqrt{113}$$

2 $A(-3, 5)$, $B(-2, -2)$ and $C(3, -7)$.
Lines are congruent if they are the same length.

For the line AB:
$(x_1, y_1) = (-3, 5), (x_2, y_2) = (-2, -2)$

$$AB = \sqrt{((-2)-(-3))^2 + ((-2)-5)^2}$$
$$= \sqrt{1^2 + (-7)^2}$$
$$= \sqrt{50}$$

For the line BC:
$(x_1, y_1) = (-2, -2), (x_2, y_2) = (3, -7)$

$$BC = \sqrt{(3-(-2))^2 + ((-7)-(-2))^2}$$
$$= \sqrt{5^2 + (-5)^2}$$
$$= \sqrt{50}$$

$AB = BC$, $\therefore$ they are congruent.

3 $P(11, -8)$, $Q(4, -3)$ and $R(7, 5)$.

Using the distance formula, and taking the unknown length as d:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

For the line PQ:

$(x_1, y_1) = (11, -8), (x_2, y_2) = (4, -3)$

$PQ = \sqrt{(4-11)^2 + ((-3)-(-8))^2}$

$= \sqrt{(-7)^2 + 5^2}$

$= \sqrt{74}$

For the line QR:

$(x_1, y_1) = (4, -3), (x_2, y_2) = (7, 5)$

$QR = \sqrt{(7-4)^2 + (5-(-3))^2}$

$= \sqrt{3^2 + 8^2}$

$= \sqrt{73}$

$PQ \neq QR$, therefore the line segments are not congruent.

4 The distance between the points $(-1, 13)$ and $(x, 9)$ is $\sqrt{65}$.

Using the distance formula, and taking the length as d:

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$65 = (x-(-1))^2 + (9-13)^2$

$65 = (x+1)^2 + (-4)^2$

$65 = x^2 + 2x + 1 + 16$

$x^2 + 2x - 48 = 0$

$(x + 8)(x - 6) = 0$

$x = -8$ or $x = 6$

5 The distance between the points $(2, y)$ and $(5, 7)$ is $3\sqrt{10}$.

Using the distance formula, and taking the length as d:

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$90 = (5-2)^2 + (7-y)^2$

$90 = 3^2 + 49 - 14y + y^2$

$y^2 - 14y - 32 = 0$

$(y - 16)(y + 2) = 0$

$y = 16$ or $y = -2$

6 a l_1: $y = 2x + 4$, gradient = 2

l_2: $6x - 3y - 9 = 0$

Rearrange line l_2 to give:

$3y = 6x - 9$

$y = 2x - 3$, gradient = 2

Lines l_1 and l_2 both have gradient 2 so they are parallel.

b Line l_3 is perpendicular to line l_1 so it has gradient $-\frac{1}{2}$.

It also passes through the point $(3, 10)$.

$y - y_1 = m(x - x_1)$

$y - 10 = -\frac{1}{2}(x - 3)$

$2y - 20 = -x + 3$

$x + 2y - 23 = 0$ is the equation of l_3

c l_2: $y = 2x - 3$

l_3: $2y = -x + 23$

Dividing through by 2:

$y = -\frac{1}{2}x + \frac{23}{2}$

At the point of intersection the two expressions for y are equal, so:

$2x - 3 = -\frac{1}{2}x + \frac{23}{2}$

6 c Then multiplying through by 2:

$4x - 6 = -x + 23$

$5x = 29$

$x = \frac{29}{5}$

Substituting $x = \frac{29}{5}$ into $y = 2x - 3$:

$y = 2\left(\frac{29}{5}\right) - 3$

$= \frac{43}{5}$

The point of intersection of the lines l_2 and l_3 is $\left(\frac{29}{5}, \frac{43}{5}\right)$.

d l_1 and l_2 are parallel so the shortest distance between them is the perpendicular distance.

l_3 is perpendicular to l_1 and therefore is perpendicular to l_2.

l_2 and l_3 intersect at $\left(\frac{29}{5}, \frac{43}{5}\right)$.

Now work out the point of intersection for lines l_1 and l_3.

l_1: $y = 2x + 4$

l_3: $y = -\frac{1}{2}x + \frac{23}{2}$

$2x + 4 = -\frac{1}{2}x + \frac{23}{2}$

$4x + 8 = -x + 23$

$5x = 15, x = 3$

When $x = 3, y = 10$

The point of intersection of the lines l_1 and l_3 is (3, 10).

Now find the distance, d, between (3, 10) and $\left(\frac{29}{5}, \frac{43}{5}\right)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{\left(\frac{29}{5} - 3\right)^2 + \left(\frac{43}{5} - 10\right)^2}$

$= \sqrt{\left(\frac{14}{5}\right)^2 + \left(-\frac{7}{5}\right)^2}$

$= \sqrt{\frac{245}{25}}$

$= \frac{1}{5}\sqrt{245}$

$= \frac{1}{5}\sqrt{49 \times 5}$

6 d $d = \frac{7\sqrt{5}}{5}$

The perpendicular distance between l_1 and l_2 is $\frac{7}{5}\sqrt{5}$.

7 Point P is on the line $y = -3x + 4$.

Its distance, d, from (0, 0) is $\sqrt{34}$.

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$34 = (x - 0)^2 + (y - 0)^2$

$34 = x^2 + y^2$

Solve $34 = x^2 + y^2$ and $y = -3x + 4$ simultaneously.

$34 = x^2 + (-3x + 4)^2$

$34 = x^2 + 9x^2 - 24x + 16$

$10x^2 - 24x - 18 = 0$

$5x^2 - 12x - 9 = 0$

$(5x + 3)(x - 3) = 0$

$x = -\frac{3}{5}$ or $x = 3$

When $x = \frac{3}{5}, y = -3\left(-\frac{3}{5}\right) + 4 = \frac{29}{5}$

When $x = 3, y = -3(3) + 4 = -5$

So P is the point $\left(-\frac{3}{5}, \frac{29}{5}\right)$ or (3, −5).

8 a In a scalene triangle, all three sides have different lengths: $AB \neq BC \neq AC$.

$A(2, 7)$, $B(5, -6)$ and $C(8, -6)$.

$AB = \sqrt{(5-2)^2 + ((-6)-7)^2}$

$= \sqrt{3^2 + (-13)^2}$

$= \sqrt{178}$

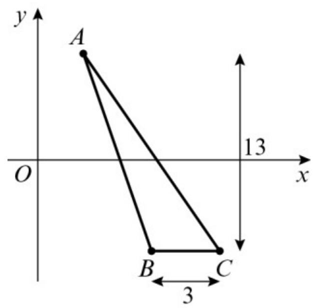
$BC = \sqrt{(8-5)^2 + ((-6)-(-6))^2}$

$= \sqrt{3^2 + 0^2}$

$= 3$

8 a $AC = \sqrt{(8-2)^2 + ((-6)-7)^2}$
$= \sqrt{6^2 + (-13)^2}$
$= \sqrt{205}$
$AB \neq BC \neq AC$ therefore ABC is a scalene triangle.

b Draw a sketch and label the points A, B and C.



Area of ΔABC $= \frac{1}{2}bh$
$= \frac{1}{2} \times 3 \times 13$
$= 19.5$

9 a l_1: $y = 7x - 3$
l_2: $4x + 3y - 41 = 0$

Substituting l_1 into l_2 gives:
$4x + 3(7x - 3) - 41 = 0$
$4x + 21x - 9 - 41 = 0$
$25x = 50$
$x = 2$

Substituting $x = 2$ into $y = 7x - 3$ gives $y = 11$.
A is the point (2, 11).

b When l_2 crosses the x-axis, $y = 0$.
So $4x + 3(0) - 41 = 0$
$4x = 41$
$x = \frac{41}{4}$
B is the point $(\frac{41}{4}, 0)$.

9 c The base of ΔAOB is $\frac{41}{4}$.
The height of ΔAOB is 11.

Area $\Delta AOB = \frac{1}{2} \times \frac{41}{4} \times 11$
$= \frac{451}{8}$

10 a l_1: $4x - 5y - 10 = 0$ intersects the x-axis at A, so $y = 0$.

$4x - 5(0) - 10 = 0$
$4x = 10$
$x = \frac{5}{2}$
A is the point $\left(\frac{5}{2}, 0\right)$.

b l_2: $4x - 2y + 20 = 0$ intersects the x-axis at B, so $y = 0$.

$4x - 2(0) + 20 = 0$
$4x = -20$
$x = -5$
B is the point (−5, 0).

c l_1: $4x = 5y + 10$, l_2: $4x = 2y - 20$

Where the lines intersect:
$5y + 10 = 2y - 20$
$3y = -30$
$y = -10$

Substituting $y = -10$ into $4x = 5y + 10$:
$4x = -40$
$x = -10$

l_1 and l_2 intersect at the point (−10, −10).

d The base of ΔABC is $7\frac{1}{2}$.
The height of the triangle is 10.
Area ΔABC $= \frac{1}{2} \times 7\frac{1}{2} \times 10$
$= \frac{75}{2}$

11 a $R(5, -2)$ and $S(9, 0)$ lie on a straight line.

The gradient, m, of the line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-2)}{9-5} = \frac{2}{4} = \frac{1}{2}$$

The equation of the line is:

$$y - y_1 = m(x - x_1)$$
$$y-(-2) = \tfrac{1}{2}(x-5)$$
$$y+2 = \tfrac{1}{2}x - \tfrac{5}{2}$$
$$y = \tfrac{1}{2}x - \tfrac{9}{2}$$

b l_2 is perpendicular to l_1 so the gradient of l_2 is -2.

The equation of the line is:

$$y - y_1 = m(x - x_1)$$
$$y-(-2) = -2(x-5)$$
$$y+2 = -2x + 10$$
$$y = -2x + 8$$

c The y-intercept for line l_2 is 8.
T is the point $(0, 8)$.

d $$RS = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(9-5)^2 + (0-(-2))^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = \sqrt{4\times 5} = 2\sqrt{5}$$

11 d $$TR = \sqrt{(5-0)^2 + ((-2)-8)^2} = \sqrt{5^2 + (-10)^2} = \sqrt{125} = 5\sqrt{5}$$

e The base of ΔRST is RS, $2\sqrt{5}$.
The height of ΔRST is RT, $5\sqrt{5}$.

$$\text{Area } \Delta RST = \tfrac{1}{2} \times 2\sqrt{5} \times 5\sqrt{5} = 25$$

12 a l_1 has gradient $m = -\tfrac{1}{4}$ and passes through the point $(-4, 14)$.

$$y - y_1 = m(x - x_1)$$
$$y - 14 = -\tfrac{1}{4}(x-(-4))$$
$$4y - 56 = -x - 4$$
$$x + 4y - 52 = 0$$

b Where l_1 crosses the y-axis, $x = 0$.
$$0 + 4y - 52 = 0$$
$$y = 13$$
A is the point $(0, 13)$.

c l_2 has gradient, $m = 3$ and passes through the point $(0, 0)$.
$y = 3x$

To find point B, substitute l_2 into l_1:
$$x + 4(3x) - 52 = 0$$
$$13x - 52 = 0$$
$$x = 4$$

Substitute $x = 4$ into $y = 3x$.
$y = 12$
B is the point $(4, 12)$.

d The base of ΔOAB is $OA = 13$.
The height of ΔOAB is the distance of B from the y-axis $= 4$
Area $\Delta OAB = \tfrac{1}{2} \times 13 \times 4 = 26$